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Shift operators and the U(N) multiplicity problem

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Abstract. A computationally effective method for decomposing r-fold tensor products of irreducible representations of U(N) in a basis-independent fashion is given. The multiplicity arising from the tensor decomposition is resolved with the eigenvalues of invariant operators chosen from the universal enveloping algebra generated by the infinitesimal operators of the dual (or complementary) representation. Shift operators which commute with the U(N) invariant operators, but not the dual invariant operators, are introduced to compute the eigenvectors and eigenvalues of the dual invariant operators algebraically. A three-fold tensor product of irreducible representations of $SU(4)$ is decomposed to illustrate the power and generality of the method.

1. Introduction

The eigenvalues and eigenvectors of some Hermitian operators can be computed algebraically using raising and lowering operators. It is of considerable interest to ask how this method can be generalized to other problems. In this paper we show that shift operators, which are like raising operators restricted to an irreducible representation space, can be used to resolve the U(N) multiplicity problem in a computationally effective way.

The motivation for constructing shift operators comes from papers by Hughes [1] in which the eigenvalues and eigenvectors of an operator X are computed in order to break the multiplicity of SO(3) representations of SU(3). Recall that the labels l and m, eigenvalues of the total and z-component of angular momentum arising from the SO(3) subgroups of SU(3), are not sufficient to specify a basis in a representation space of SU(3). The eigenvalues of an additional operator X in the enveloping algebra of SU(3) that commutes with SO(3) are needed to specify a basis uniquely. Hughes introduced shift operators that commuted with the z-component of angular momentum and acted like raising and lowering operators on L^2. Hughes used these shift operators to calculate the eigenvalues of X which are irrational.

We will reformulate Hughes's problem using the notion of dual representations. In order to do this we begin by making some definitions.

Definition 1.1. Let G and G' be two groups. Let R and L be representations of G and G', respectively on a Hilbert space $\mathcal{F}$ such that the two actions commute. Assume that the representations R of G and L of G' on $\mathcal{F}$ are completely reducible, i.e. are the direct sum of irreducible representations; then we say that R and L are *dual* (or complementary) if the spectral decomposition of R determines that of L completely and vice versa.

If we consider the joint action $L \otimes R$ of $G' \times G$ on $\mathcal{F}$, then L and R are dual if

$$\mathcal{F} = \sum_{\chi} \oplus \mathcal{I}^{\chi}$$

and $L \otimes R|_{\mathcal{I}^{\chi}}$ is irreducible for each χ, where χ is an index that characterizes both an irreducible representation of G and G'; $\mathcal{I}^{\chi}$ is the isotypic component of the representation of G (respectively G') in $\mathcal{F}$ (i.e. the largest G-invariant (respectively G'-invariant) subspace of $\mathcal{F}$ which contains all irreducible representations spaces that are equivalent to the one characterized by χ) [2]. This generalizes the notion of complementary groups introduced by Moshinsky and Quesne [3] and the notion of dual pairs by Howe [4].

The theory of dual representations will play an important role in resolving the multiplicity problem for a group action restricted to a subgroup. Before we reformulate Hughes's problem and state the general multiplicity problem, we will define invariant operators of a group action restricted to a subgroup, and the main topic of this paper, shift operators.

Definition 1.2. Let G be a Lie group which acts on a Hilbert space $\mathcal{F}$. The infinitesimal operators of this group action generate a Lie algebra. Let $\mathcal{U}(G)$ be the universal enveloping algebra. Let H be a subgroup of G, and restrict the action of G on $\mathcal{F}$ to H. An operator $X \in \mathcal{U}(G)$ is an *invariant operator of G restricted to H* if $[X, h] = 0$ for all infinitesimal operators h of the H action. We will denote the set of invariant operators of G restricted to H by $C_{\mathcal{U}(G)}(H)$.

As we will see shift operators are operators which map an irreducible representation space into a given reducible representation space which intertwine and satisfy a given commutation relation on the irreducible representation space.

Recall that if V is a representation space of G, then V is called a G-module, and if V and W are G-modules, $\mathrm{Hom}_G(V, W)$ denotes the vector space of all intertwining operators from the G-modules V into the G-modules W.

Definition 1.3. Let G be a Lie group which acts on a Hilbert space $\mathcal{F}$. The infinitesimal operators of this group generate a Lie algebra. Let $\mathcal{U}(G)$ be the universal enveloping algebra generated by this Lie algebra. Let $W \subset \mathcal{F}$ be an irreducible G-submodule. Let H be a subgroup of G and let V be an irreducible H-module. Let $X_1, \ldots, X_q$ be a commuting family of Hermitian operators in $C_{\mathcal{U}(G)}(H)$. If $\tilde{\Lambda} \in \mathrm{Hom}_H(V, W)$, then $\tilde{\Lambda}$ is a *shift operator* if $[X_i, \tilde{\Lambda}]f = \lambda_i(V)\tilde{\Lambda}f \ \forall i = 1, \ldots, q, f \in V$, where $\lambda_i(V)$ is a scalar which depends only on V and X_i.

We call $\tilde{\Lambda}$ a shift operator rather than a raising operator since the commutation relation is only valid for elements f in V. We will also see that $\tilde{\Lambda}$ shifts the eigenvalues of the invariant operators $X_1, \ldots, X_q$.

We will now reformulate Hughes's problem in the language of dual representations, invariant operators and shift operators.

For Hughes's problem the Hilbert space $\mathcal{F}$ can be chosen to be the Fock space $\mathcal{F}(\mathbb{C}^{2\times 3})$ which consists of all holomorphic square integrable functions with respect to a Gaussian measure (see section 2). The group G in his problem is SU(3) and the subgroup H is SO(3). By the theory of dual pairs the group dual to $G = \mathrm{SU}(3)$ is $G' = \mathrm{SU}(2)$ and the group dual to $H = \mathrm{SO}(3)$ is $H' = \mathrm{Sp}(4, \mathbb{R})$. Let W be an irreducible G-module and V an irreducible H-module. It is known that $C_{\mathcal{U}(\mathrm{Sp}(4,\mathbb{R}))}(\mathrm{SU}(2)) = C_{\mathcal{U}(\mathrm{SU}(3))}(\mathrm{SO}(3))$. If the multiplicity of V in W is μ, then Hughes's problem reduces to finding a commuting family of operators in $C_{\mathcal{U}(\mathrm{Sp}(4,\mathbb{R}))}(\mathrm{SU}(2))$ whose eigenvalues can be used to break the multiplicity, and μ shift operators in $\mathcal{U}(\mathrm{Sp}(4,\mathbb{R}))$ which form an orthogonal basis for $\mathrm{Hom}_{\mathrm{SO}(3)}(V, W)$.

The reformulation of Hughes's problem in terms of dual representations suggests how to formulate the general multiplicity problem of a group action restricted to a subgroup action. Let G be a group which acts on a Hilbert space $\mathcal{F}$ and let W be an irreducible G-module. Let H be a subgroup of G, and let V be an irreducible H-module. If we restrict the action of G to H, then a state in W can be labelled by $|\chi^G, \chi^H, \mu^H, \eta\rangle$, where χ^G labels the space W, χ^H labels V, μ^H are the eigenvalues of commuting Lie algebra elements of H, and η is a multiplicity label, which will be chosen to be eigenvalues of commuting invariant operators. If G' is the action dual to G on $\mathcal{F}$ and H' is the action dual to H on $\mathcal{F}$, then we have the following diagram:

$$\begin{array}{ccc} G' \searrow & & \swarrow G \\ \cap & \mathcal{F} & \cup \\ H' \nearrow & & \nwarrow H \end{array}$$

Note that when we restrict the action of G to H that the dual action gets larger. By the theory of dual representations there is an irreducible G'-module, $W_{G'}$, which is labelled by χ^G and there is an irreducible H'-module, $W_{H'}$, which is labelled by χ^H. If we assume $C_{\mathcal{U}(H')}(G') = C_{\mathcal{U}(G)}(H)$, then the multiplicity problem reduces to finding a commuting family of Hermitian operators, $X_1, \ldots, X_q$ in $C_{\mathcal{U}(H')}(G')$ whose eigenvalues can be used to break the multiplicity of V in W and shift operators which form an orthogonal basis for $\mathrm{Hom}_H(V, W)$.

We will now show how this general setup applies to the decomposition of r-fold tensor products of U(N). Since the finite-dimensional irreducible representations of U(N) and GL(N, $\mathbb{C}$) are the same, we will, in general, work with GL(N, $\mathbb{C}$).

We want to decompose the tensor product $V^{(M_{(1)})} \otimes \cdots \otimes V^{(M_{(r)})}$ of arbitrary irreducible representations of GL(N, $\mathbb{C}$), where $(M_{(1)}), \ldots, (M_{(r)})$ are each dominant N-tuples of integers. We begin by forming an n-tuple of integers (m) which is obtained by deleting all zeros from $(M_{(1)}), \ldots, (M_{(r)})$. Our Hilbert space $\mathcal{F}$ is the Fock space $\mathcal{F}(\mathbb{C}^{n\times N})$ (see section 2). The group G which acts on $\mathcal{F}$ by right translation is

$$G = \underbrace{\mathrm{GL}(N,\mathbb{C}) \times \cdots \times \mathrm{GL}(N,\mathbb{C})}_{r}$$

and the action dual to G is

$$G' = \mathrm{GL}(p_i,\mathbb{C}) \times \cdots \times \mathrm{GL}(p_r,\mathbb{C})$$

where p_i is the number of non-zero entries of $(M_{(i)})$. The space $W = V^{(M_{(1)})} \otimes \cdots \otimes V^{(M_{(r)})}$ is an irreducible G-module and is labelled by (M). We restrict G to its diagonal subgroup which we denote by H. Since H is isomorphic to GL(N, $\mathbb{C}$), its dual action is $H' = GL(n, \mathbb{C})$. Let $V = V^{(m)}$ be an irreducible GL(N, $\mathbb{C}$)-module and suppose that V occurs in W with multiplicity μ. We have shown that $C_{\mathcal{U}(G)}(H) = C_{\mathcal{U}(H')}(G')$ [5], so our problem is to find a commuting family of invariant operators in $C_{\mathcal{U}(H')}(G')$ which breaks the multiplicity of V in W and μ shift operators in $\mathcal{U}(H')$ which form an orthogonal basis for $\mathrm{Hom}_H(V, W)$.

In section 2 of this paper we will review our results concerning the decomposition of arbitrary representations of U(N). The main tools to carry out this decomposition are a

Fock space in $n \times N$ complex variables which is the carrier space for tensor products, a Frobenius reciprocity theorem which provides a method to compute the multiplicity, and the theory of dual representations which will be used to construct $C_{U(H')}(G')$.

In section 3 we prove the existence of shift operators for the tensor product decomposition of arbitrary irreducible representations of U(N). The theorem shows that the eigenvalues of our invariant operators depend only on (m). Further it gives us an algorithm to generate the shift operators. The paper closes with a long example.

2. The decomposition of arbitrary tensor products of representations of U(N)

Let

$$\mathcal{F}(\mathbb{C}^{n\times N}) = \left\{ f : \mathbb{C}^{n\times N} \to \mathbb{C} \mid f \text{ holomorphic}, \int_{\mathbb{C}^{n\times N}} |f(Z)|^2 \, \mathrm{d}\mu(Z) < \infty \right\}$$

where $z = (z_{\alpha j})$ with $z_{\alpha j} = x_{\alpha j} + \mathrm{i} y_{\alpha j}$; $1 \leqslant \alpha \leqslant n$, $1 \leqslant j \leqslant N$.

$$\mathrm{d}\mu(Z) = \frac{1}{\pi^{nN}} \exp(-\operatorname{tr}(ZZ^{\dagger})) \qquad \text{and} \qquad \mathrm{d}Z = \prod_{\alpha=1}^{n} \prod_{j=1}^{N} \mathrm{d}x_{\alpha j} \, \mathrm{d}y_{\alpha j}.$$

It is clear that $\mathcal{F}(\mathbb{C}^{n\times N})$ is a Hilbert space with respect to the inner product:

$$(f, g) = \int_{\mathbb{C}^{n\times N}} f(Z) \overline{g(Z)} \, \mathrm{d}\mu(Z). \tag{2.1}$$

Let $P(\mathbb{C}^{n\times N}) = \{p : \mathbb{C}^{n\times N} \to \mathbb{C} \mid p \text{ polynomial}\}$, then it is clear that P is dense in $\mathcal{F}$. If we endow P with the 'differentiation' inner product given by

$$\langle p_1, p_2 \rangle = p_1(D) \overline{p_2(\overline{Z})} |_{Z=0} \tag{2.2}$$

where $p_1(D)$ is obtained by replacing $z_{\alpha j}$ by $(\partial/\partial z_{\alpha j})$, then it can be shown [5] that $\langle \cdot, \cdot \rangle = (\cdot, \cdot)|_{P(\mathbb{C}^{n\times N})}$. Computationally, this result is very important since it reduces the inner product to differentiation of polynomials which is easily done on a computer. In fact it can be shown that the set of all monomials in $P(\mathbb{C}^{n\times N})$ are orthogonal, and the norm of a monomial is the product of the factorials of its exponents, so the inner product of two polynomials further reduces to a weighted dot product.

Let $H' = \mathrm{GL}(n, \mathbb{C})$ act on $\mathbb{C}^{n\times N}$ to the left and $H = \mathrm{GL}(N, \mathbb{C})$ act on $\mathbb{C}^{n\times N}$ to the right, then these actions induce actions of $\mathcal{F}(\mathbb{C}^{n\times N})$ given by

$$[L(h')f](Z) = f((h')^{-1}Z) \qquad \forall (Z, h') \in \mathbb{C}^{n\times N} \times H' \qquad f \in \mathcal{F}$$
$$[R(g)f](Z) = f(Zg) \qquad \forall (Z, g) \in \mathbb{C}^{n\times N} \times H \qquad f \in \mathcal{F}.$$

Let $(M) = (M_1, \ldots, M_n)$ be an n-tuple of non-negative integers and define

$$P^{(M)} = \left\{ p \in \mathcal{F} | p \text{ polynomial}, p(\mathrm{d}Z) = d_{11}^{M_1} \ldots d_{nn}^{M_n} p(Z), \forall d = \begin{pmatrix} d_{11} & & 0 \\ & \ddots & \\ 0 & & d_{nn} \end{pmatrix} \in D_n \right\}. \tag{2.3}$$

The elements of $P^{(M)}$ are said to transform covariantly with respect to the left diagonal subgroup D_n; further $\mathcal{F}(\mathbb{C}^{n\times N}) = \sum_{(M)} \oplus P^{(M)}$ (see [5, 6]).

Let $(m) = (m_1, \ldots, m_n)$ be dominant, i.e. $m_1 \geqslant m_2 \geqslant \cdots \geqslant 0$ and define

$$V^{(m)} = \left\{ p \in \mathcal{F} | p \text{ polynomial}, p(bZ) = b_{11}^{m_1} \ldots b_{nn}^{m_n} p(Z), \forall b = \begin{pmatrix} b_{11} & & 0 \\ & \ddots & \\ * & & b_{nn} \end{pmatrix} \in B_n \right\}. \tag{2.4}$$

The elements of $V^{(m)}$ are said to transform covariantly with respect to the lower triangular subgroup B_n. Furthermore, $V^{(m)}$ is an irreducible representation space of G, and $P^{(M)} \simeq V^{(M_1,0,\ldots,0)} \otimes \cdots \otimes V^{(M_n,0,\ldots,0)}$ (see [5]).

In general, the space $P^{(M)}$ is not invariant under the action of L so we introduce $P^{|M|} = \{p \in \mathcal{F} \mid p \text{ polynomial}, \ p(\lambda Z) = \lambda^{|M|} p(Z)\}$ where $\lambda \in \mathbb{C}$ and $|M| = \sum_{i=1}^{n} M_i$. It is clear that $P^{(M)} \subset P^{|M|}$, the actions of L and R commute on $P^{|M|}$ and leave $P^{|M|}$ invariant.

Next we define the isotypic component, $\mathcal{I}(V^{(m)})$ of $V^{(m)} \subset \mathcal{F}$, to be the sum of all submodules in $\mathcal{F}$ which are isomorphic to $V^{(m)}$. If $P^{|M|}$ contains a submodule isomorphic to $V^{(m)}$, then $\mathcal{I}(V^{(m)}) \subset P^{|M|}$ (see [5]).

We will now show how to decompose $V^{(M_{(1)})} \otimes \cdots \otimes V^{(M_{(r)})}$ where $(M_{(i)}) = (M_{i1}, \ldots, M_{iN})$ is the signature of an arbitrary irreducible representation of G. We begin by forming an n-tuple of integers: $(M) = (M_1, \ldots, M_{p_1}, M_{p_1+1}, \ldots, M_{p_1+p_2}, \ldots, M_{p_1+\cdots+p_r})$ where $M_1, \ldots, M_{p_1}$ are the p_1 non-zero entries of $(M_{(1)})$, $M_{p_1+1}, \ldots, M_{p_1+p_2}$ are the p_2 non-zero entries of $(M_{(2)})$, etc, and $p_1 + \cdots + p_r = n$.

It is clear that H' contains the subgroup $G' = \mathrm{GL}(p_1, \mathbb{C}) \times \cdots \times \mathrm{GL}(p_r, \mathbb{C})$ which consists of all elements of $\mathrm{GL}(n, \mathbb{C})$ of the form

$$\begin{pmatrix} g_1' & & 0 \\ & \ddots & \\ 0 & & g_r' \end{pmatrix}$$

where $g_i' \in \mathrm{GL}(p_i, \mathbb{C})$ $\forall i = 1, \ldots, r$. Let B_{p_i} denote the lower triangular subgroup of $\mathrm{GL}(p_i, \mathbb{C})$ and define

$$W^{(M)} = \left\{ p \in \mathcal{F} \mid p \text{ polynomial}, \ p\left(\begin{pmatrix} b_{p_1} & & 0 \\ & \ddots & \\ 0 & & b_{p_r} \end{pmatrix} Z \right) = b_{11}^{M_1} \ldots b_{nn}^{M_n} p(Z), \right.$$
$$\left. \text{where } b_{p_i} \in B_{p_i} \right\}. \tag{2.5}$$

Then $W^{(M)} \subset P^{(M)}$ and $H^{(M)} \approx V^{(M_{(1)})} \otimes \cdots \otimes V^{(M_{(r)})}$ (see [6]).

Let $L_{\alpha\beta}$ denote the infinitesimal operators of L which have the form

$$L_{\alpha\beta} = \sum_{k=1}^{N} z_{\alpha k} \frac{\partial}{\partial z_{\beta k}}. \tag{2.6}$$

Among these operators are the infinitesimal operators of $G' : L_{\alpha_{p_i}\beta_{p_i}}$ where

$$\left(\sum_{w=1}^{i-1} p_w \right) + 1 \leqslant \alpha_{p_i}, \beta_{p_i} \leqslant \sum_{w=1}^{i} p_w \qquad \forall i = 1, \ldots, r.$$

The operators $L_{\alpha_{p_i}\beta_{p_i}}$' with $\alpha_{p_i} < \beta_{p_i}$ lead us to the following characterization of $W^{(M)}$: $W^{(M)} = \{p \in P^{(M)} \mid L_{\alpha_{p_i}\beta_{p_i}} p = 0, \forall \alpha_{p_i} < \beta_{p_i}, i = 1, \ldots, r\}$. Hence $W^{(M)}$ is the set of all polynomials in $P^{(M)}$ which are simultaneously annihilated by the operators $L_{\alpha_{p_i}\beta_{p_i}}$ with $\alpha_{p_i} < \beta_{p_i}$.

We have now shown how all irreducible representations of H and how tensor products of arbitrary representations of H can be concretely realized as polynomials. We will now show how we can compute the multiplicity of $V^{(m)}$ in $P^{(M)}$. This number together with our characterization of $W^{(M)}$ will lead us to the multiplicity of $V^{(m)}$ in $W^{(M)}$. Furthermore, we will see that the invariant operators that we use to break the multiplicity commute with the set of operators $L_{\alpha_{p_i}\beta_{p_i}}$ (without the condition $\alpha_{p_i} < \beta_{p_i}$) which means that we can diagonalize the invariant operators on $P^{(M)}$ first and then project the eigenvectors into $W^{(M)}$.

In order to calculate the multiplicity of $V^{(m)}$ in $P^{(M)}$ we use the following Frobenius reciprocity theorem (see [5]).

Theorem 2.1.

(a) If $n \leqslant N$ then the frequency of occurrence of the irreducible representation of $\mathrm{GL}(N, \mathbb{C})$ with signature $(m_1, \ldots, m_N)$ in $P^{(M)}$ is equal to the dimension of the weight space $(M_1, \ldots, M_n, 0, \ldots, 0)$ in $V^{(m_1, \ldots, m_N)}$.

(b) If $n > N$ then the frequency of occurrence of the irreducible representation of $\mathrm{GL}(N, \mathbb{C})$ with signature $(m_1, \ldots, m_N)$ in $P^{(M)}$ is equal to the dimension of the weight space $(M_1, \ldots, M_N)$ in the representation space $V^{(M_1, \ldots, M_N)}$ of $GL(n, \mathbb{C})$.

Concretely, we can calculate the number of times $V^{(m)}$ occurs in $P^{(M)}$, $n(V^{(m)}, P^{(M)})$, with the help of Gelfand tableaux $\substack{(m)\\ [t]}$ of weight (M). Recall that if

$$\begin{matrix}(m)\\ [t]\end{matrix} = \begin{pmatrix} m_1 \; m_2 \; \ldots \; m_n \\ m_{1,n-1} \; \cdots \; m_{n-1,n-1} \\ \ddots \quad \iddots \\ m_{1,2} \;\; m_{2,2} \\ m_{1,1} \end{pmatrix}$$

is a Gelfand tableau, then $\substack{(m)\\ [t]}$ satisfies the betweeness relations $m_{i,k} \geqslant m_{i,k-1} \geqslant m_{i+1,k}$ $\forall k = 2, \ldots, n$, $\forall i = 1, \ldots, k-1$, then $\substack{(m)\\ [t]}$ has weight (M) if and only if $\sum_{k=1}^{i} m_{i,k} = \sum_{k=1}^{i} M_k$, $\forall i = 1, \ldots, n$ (see [7]). We have written a computer program to generate the set of all Gelfand tableaux $\substack{(m)\\ [t]}$ with weight (M) [8].

In order to find maps from $V^{(m)}$ to $P^{(M)}$, we consider the infinitesimal operators $L_{\alpha\beta}$, $1 \leqslant \alpha, \beta \leqslant n$, in (2.6). These operators form a basis for a Lie algebra of H' with commutation relations:

$$[L_{\alpha\beta}, L_{\gamma\eta}] = L_{\alpha\eta}\delta_{\beta\gamma} - L_{\gamma\beta}\delta_{\alpha\eta} \qquad 1 \leqslant \alpha, \beta, \gamma, \eta \leqslant n \tag{2.7}$$

and generate a universal enveloping algebra $\mathcal{U}(H')$ of right invariant differential operators which act on $\mathcal{F}$. Moreover, by the Poincaré–Birkhoff–Witt theorem the ordered monomial in $L_{\alpha\beta}$ forms a basis for $\mathcal{U}(H')$. Suppose that $n(V^{(m)}, P^{(M)}) = \mu$, then, as a consequence of Burnside's theorem (see [7]), there exist μ linearly independent elements in $\mathcal{U}(H')$, $p_1(L_{\alpha\beta}), \ldots, p_\mu(L_{\alpha\beta})$ which form a basis for the vector space $\mathrm{Hom}_H(V^{(m)}, P^{(M)})$. We will show by example in section 4 how the set of Gelfand tableaux $\binom{(m)}{[t]}$ with weight (M) can be used to generate a basis for $\mathrm{Hom}_H(V^{(m)}, P^{(M)})$.

If $f \in V^{(m)}$ then, in general, the polynomials $p_1(L_{\alpha\beta})f, \ldots, p_\mu(L_{\alpha\beta})f \in P^{(M)}$ are not orthogonal. In order to obtain an orthogonal direct sum decomposition of $\mathcal{I}(V^{(m)}) \cap W^{(M)}$, we must find operators which commute with the action of G' or, equivalently, with the operators $L_{\alpha_{p_i}\beta_{p_i}}$ (without the condition $\alpha_{p_i} < \beta_{p_i}$), and that decompose $\mathcal{I}(V^{(m)}) \cap W^{(M)}$ into distinct eigenspaces.

To carry out this decomposition we concentrate on the action of G' and its right dual action on $\mathcal{F}(\mathbb{C}^{n\times N})$. Let $Z \in \mathbb{C}^{n\times N}$ and write Z in block form as

$$Z = \begin{bmatrix} Z_1 \\ \vdots \\ Z_r \end{bmatrix}$$

where each Z_i is a $p_i \times N$ matrix, $1 \leqslant i \leqslant r$.

The action of G' on Z is of the form:

$$(g_1', \ldots, g_r') \to \begin{bmatrix} g_1' Z_1 \\ \vdots \\ g_r' Z_r \end{bmatrix} \qquad \forall (g_1', \ldots, g_r') \in G'.$$

Its dual action is therefore

$$(g_1, \ldots, g_r) \to \begin{bmatrix} Z_1 g_1 \\ \vdots \\ Z_r g_r \end{bmatrix} \qquad \forall (g_1, \ldots, g_r) \in G.$$

By the theory of dual representations, to find operators in $\mathcal{U}(H')$ which commute with the action of G' is equivalent to finding operators in $\mathcal{U}(G)$ which commute with the action of the diagonal subgroup H. Set

$$R_{kl}^{(p_i)} = \sum_{q=1}^{p_i} z_{qk} \frac{\partial}{\partial z_{ql}} \qquad 1 \leqslant k, l \leqslant N$$

and let $[R^{(p_i)}]$ denote the matrix $(R_{kl}^{(p_i)})$.

Set $[L] = (L_{\alpha\beta})$, $1 \leqslant \alpha, \beta \leqslant n$ and write the matrix $[L]$ in block form as

$$[L] = \begin{pmatrix} [L]_{11} & \ldots & [L]_{1r} \\ \vdots & & \vdots \\ [L]_{r1} & \ldots & [L]_{rr} \end{pmatrix}$$

where each $[L]_{uv}$ is a $p_u \times p_v$ matrix, $1 \leqslant u, v \leqslant r$. The following theorems give us the explicit form of the operators that we are looking for (see [9]).

Theorem 2.2. In the universal enveloping algebra $\mathcal{U}(H')$, the elements of the form $\mathrm{Tr}([L]_{u_1u_2}[L]_{u_2u_3}\ldots[L]_{u_qu_1})$ generate a subalgebra of Hermitian differential operators which commute with the action of G' on $\mathcal{F}(\mathbb{C}^{n\times N})$.

Theorem 2.3. The differential operators of form $\mathrm{Tr}([R^{(p_i)}]^{d_1}\ldots[R^{(p_i)}]^{d_r})$, where d_i are integers $\geqslant 0$, $\forall i = 1, \ldots, r$ generate the same algebra of G'-invariant differential operators as those defined in theorem 2.2.

The operators defined in theorem 2.2 are Hermitian. Since they are Hermitian, their eigenvalues are real and their eigenvectors are orthogonal, and we may diagonalize them and use their eigenvectors to decompose $\mathcal{I}(V^{(m)}) \cap W^{(M)}$. For computational purposes the operators $\mathrm{Tr}([L]_{u_1u_2}[L]_{u_2u_3}\ldots[L]_{u_qu_1})$ are more convenient than the operators $\mathrm{Tr}((R^{(p_i)})^{d_1}\ldots(R^{(p_i)})^{d_r})$.

Observe that we may write

$$\mathrm{Tr}([L]_{u_1u_2}[L]_{u_2u_3}\ldots[L]_{u_qu_1}) = \sum_{\alpha_1=l_1}^{h_1}\cdots\sum_{\alpha_q=l_q}^{h_q} L_{\alpha_1\alpha_2}L_{\alpha_2\alpha_3}\ldots L_{\alpha_q\alpha_1}$$

and $l_1,\ldots,l_q$, $h_1,\ldots,h_q$ depend on $p_1,\ldots,p_r$. The right operators R_{kl} which make up $\mathrm{Tr}((R^{(p_i)})^{d_1})\ldots(R^{(p_i)})^{d_r})$ are of the form

$$R_{kl}^{(p_i)} = \sum_{q=1}^{p_i} z_{qk}\frac{\partial}{\partial z_{ql}}.$$

This shows that the operators $\mathrm{Tr}([L]_{u_1u_2}[L]_{u_2u_3}\ldots[L]_{u_qu_1})$ are more convenient since $p_1,\ldots,p_r$ only appear in the limits of summation, whereas the definition of $R_{kl}^{(p_i)}$ depends explicitly on p_i. Furthermore, in the special case where $l_i = 1$ and $h_i = q$, $\forall i = 1,\ldots,q$, the operator $\mathrm{Tr}([L]_{u_1u_2}[L]_{u_2u_3}\ldots[L]_{u_qu_1})$ is the qth-order Casimir operator of $\mathrm{GL}(n,\mathbb{C})$. It is well known that the eigenvalues of these Casimir operators are integers [7], whereas the eigenvalues of $\mathrm{Tr}([L]_{u_1u_2}[L]_{u_2u_3}\ldots[L]_{u_qu_1})$ can be irrational [8]. Our procedure for diagonalizing the invariant commuting operators $X_1\ldots X_q$ makes use of shift operators defined in the next section.

3. Shift operators

Shift operators are like raising operators but restricted to a definite representation space. We begin this section with the following theorem concerning shift operators defined in definition 1.3.

Theorem 3.3. Suppose that the multiplicity of $V^{(m)}$ in $W^{(M)}$ is μ and let $\Lambda_1,\ldots,\Lambda_\mu$ be μ linearly independent intertwining operators consisting solely of lexigraphically ordered lowering operators in $\mathcal{U}(H')$ which span $\mathrm{Hom}_H(V^{(m)}, W^{(M)})$. Let $X_1,\ldots,X_q$ be a commuting family of invariant operators in $C_{\mathcal{U}(H')}(G') = C_{\mathcal{U}(G)}(H)$ which break the multiplicity of $V^{(m)}$ in $W^{(M)}$. Then there exist μ shift operators $\tilde{\Lambda}_1,\ldots,\tilde{\Lambda}_\mu$ such that $\tilde{\Lambda}_1 f,\ldots,\tilde{\Lambda}_\mu f$, are linear combinations of $\Lambda_1 f,\ldots,\Lambda_\mu f$ whose coefficients are functions of (m).

The proof of theorem 3.3 is essentially an application of the Poincaré–Birkhoff–Witt theorem and is given in the thesis of Wills [10]. We have written computer programs [8] to generate the matrices needed to obtain the shift operators using the algorithm described in the proof of theorem 3.3. That we have a computationally effective method can be seen in the following example.

4. Example

4.1. Calculation of the shift operators associated with the irreducible representation (3,2,1,1) of GL(4, $\mathbb{C}$) in the tensor product $V^{(2,1,0,0)} \otimes V^{(2,1,0,0)} \otimes V^{(1,0,0,0)}$.

According to our procedures for breaking multiplicity, we consider the Fock space $\mathcal{F}(\mathbb{C}^{5\times 4})$, ($p_1 = p_2 = 2, p_3 = 1$), which contains the GL(4, $\mathbb{C}$)-module $P^{(2,1,2,1,1)}(\mathbb{C}^{5\times 4})$. The module $P^{(2,1,2,1,1)}(\mathbb{C}^{5\times 4})$, in turn, contains the submodule $W^{(2,1,2,1,1)}(\mathbb{C}^{5\times 4})$ which is isomorphic to $V^{(2,1,0,0)} \otimes V^{(2,1,0,0)} \otimes V^{(1,0,0,0)}$. The submodule $W^{(2,1,2,1,1)}$ consists of all polynomial functions in $P^{(2,1,2,1,1)}$ which are simultaneously annihilated by the operators $L_{12} = \sum_{k=1}^{4} z_{1k}(\partial/\partial z_{2k})$ and $L_{34} = \sum_{k=1}^{4} z_{3k}(\partial/\partial z_{4k})$. By theorem 2.1 the number of times that $V^{(3,2,1,1)}$ occurs in $P^{(2,1,2,1,1)}$ is equal to the dimension of the weight space (2,1,2,1,1) in $V^{(3,2,1,1,0)}$. Recall that this dimension can be found by generating all Gelfand tableaux $\begin{smallmatrix}(3\ 2\ 1\ 1\ 0)\\ [t]\end{smallmatrix}$ with weight (2,1,2,1,1). Consider the Gelfand tableau

$$\begin{matrix}(3\ 2\ 1\ 1\ 0)\\ [t]\end{matrix} = \begin{pmatrix} 3 & 2 & 1 & 1 & 0 \\ & i_1 & i_2 & i_3 & i_4 \\ & j_1 & j_2 & j_3 & \\ & & k_1 & k_2 & \\ & & l & & \end{pmatrix}.$$

A basis element labelled by this tableau has weight (2,1,2,1,1) if and only if $i_1+i_2+i_3+i_4 = 6$, $j_1 + j_2 + j_3 = 5$, $k_1 + k_2 = 3$, $l = 2$ and the betweeness relations of the Gelfand tableau are satisfied. This leads to the six possible tableaux:

$$\begin{pmatrix} 3\,2\,1\,1\,0 \\ 3\,2\,1\,0 \\ 3\,2\,0 \\ 3\,0 \\ 2 \end{pmatrix} \quad \begin{pmatrix} 3\,2\,1\,1\,0 \\ 3\,2\,1\,0 \\ 3\,2\,0 \\ 2\,1 \\ 2 \end{pmatrix} \quad \begin{pmatrix} 3\,2\,1\,1\,0 \\ 3\,2\,1\,0 \\ 3\,1\,1 \\ 2\,1 \\ 2 \end{pmatrix}$$
$$\begin{pmatrix} 3\,2\,1\,1\,0 \\ 3\,1\,1\,1 \\ 3\,1\,1 \\ 2\,1 \\ 2 \end{pmatrix} \quad \begin{pmatrix} 3\,2\,1\,1\,0 \\ 2\,2\,1\,1 \\ 2\,2\,1 \\ 2\,1 \\ 2 \end{pmatrix} \quad \begin{pmatrix} 3\,2\,1\,1\,0 \\ 3\,2\,1\,0 \\ 2\,2\,1 \\ 2\,1 \\ 2 \end{pmatrix}.$$

Hence $V^{(3,2,1,1)}$ occurs in $P^{(2,1,2,1,1)}$ with multiplicity 6. We will now show how to use these Gelfand tableaux to generate a basis for $\mathrm{Hom}_{\mathrm{GL}(4,\mathbb{C})}(V^{(3,2,1,1,0)}, P^{(2,1,2,1,1)})$. For each tableau we begin by forming a 5×5 matrix

$$\begin{pmatrix} 3 & 2 & 1 & 1 & 0 \\ i_1 & i_2 & i_3 & i_4 & 1 \\ j_1 & j_2 & j_3 & 1 & 1 \\ k_1 & k_2 & 2 & 1 & 1 \\ 2 & 1 & 2 & 1 & 1 \end{pmatrix}$$

and associate with this matrix the chain of spaces

$$V^{(3,2,1,1,0)} \to P^{(i_1,i_2,i_3,i_4,1)} \to P^{(j_1,j_2,j_3,1,1)} \to P^{(k_1,k_2,2,1,1)} \to P^{(2,1,2,1,1)}.$$

Since the action of the lowering operator $L_{\alpha\beta}$ on $P^{(M)}$ adds 1 to the α-slot and subtracts 1 from the β-slot, it is straightforward to find products of lowering operators which map

$V^{(3,2,1,1,0)}$ into $P^{(2,1,2,1,1)}$ and follow the chain above. For example, to find the map from $V^{(3,2,1,1,0)}$ to $P^{(2,1,2,1,1)}$ corresponding to the tableau

$$\begin{pmatrix} 3\,2\,1\,1\,0 \\ 3\,2\,1\,0 \\ 2\,2\,1 \\ 2\,1 \\ 2 \end{pmatrix}$$

we first form the matrix

$$\begin{pmatrix} 3\,2\,1\,1\,0 \\ 3\,2\,1\,0\,1 \\ 3\,2\,0\,1\,1 \\ 3\,0\,2\,1\,1 \\ 2\,1\,2\,1\,1 \end{pmatrix}$$

and consider the chain of spaces: $V^{(3,2,1,1,0)} \to P^{(3,2,1,0,1)} \to P^{(3,2,0,1,1)} \to P^{(3,0,2,1,1)} \to P^{(2,1,2,1,1)}$. It is clear that the operator $\Lambda_1 = L_{21}L_{32}^2L_{43}L_{54}$ maps $V^{(3,2,1,1,0)}$ into $P^{(2,1,2,1,1)}$ and follows the above chain. Similarly the operators corresponding to the other tableaux are given by $\Lambda_2 = L_{31}L_{32}L_{43}L_{54}$, $\Lambda_3 = L_{31}L_{42}L_{54}$, $\Lambda_4 = L_{31}L_{52}$, $\Lambda_5 = L_{32}L_{51}$, and $\Lambda_6 = L_{32}L_{41}L_{54}$. The operators $\Lambda_1, \ldots, \Lambda_6$ form a basis for $\mathrm{Hom}_{\mathrm{GL}(4,\mathbb{C})}(V^{(3,2,1,1,0)}, P^{(2,1,2,1,1)})$. Since $p_1 = 2$, $p_2 = 2$, $p_3 = 1$, we consider the matrix:

$$[L] = \begin{pmatrix} [L]_{11} & [L]_{12} & [L]_{13} \\ [L]_{21} & [L]_{22} & [L]_{23} \\ [L]_{31} & [L]_{32} & [L]_{33} \end{pmatrix}$$

where

$$[L]_{11} = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} \qquad [L]_{12} = \begin{pmatrix} L_{13}L_{14} \\ L_{23}L_{24} \end{pmatrix} \qquad [L]_{13} = \begin{pmatrix} L_{15} \\ L_{25} \end{pmatrix}$$

$$[L]_{21} = \begin{pmatrix} L_{31} & L_{32} \\ L_{41} & L_{42} \end{pmatrix} \qquad [L]_{22} = \begin{pmatrix} L_{33} & L_{34} \\ L_{43} & L_{44} \end{pmatrix} \qquad [L]_{23} = \begin{pmatrix} L_{35} \\ L_{45} \end{pmatrix}$$

$$[L]_{31} = (L_{51} \quad L_{52}) \qquad [L]_{32} = (L_{53} \quad L_{54}) \qquad [L]_{33} = (L_{55}).$$

By theorem 2.2 the invariant operator:

$$\begin{aligned} X &= \mathrm{Tr}([L]_{12}[L]_{21}[L]_{12}[L]_{21}) \\ &= \mathrm{Tr}(([L]_{12}[L]_{21})^2) \\ &= \sum_{i,j=1}^{2} \sum_{r,s=3}^{4} L_{ir}L_{rj}L_{js}L_{si} \end{aligned}$$

commutes with the action of the subgroup $G' = \mathrm{GL}(2,\mathbb{C}) \times \mathrm{GL}(2,\mathbb{C}) \times \mathrm{GL}(1,\mathbb{C})$ of $H' = \mathrm{GL}(5,\mathbb{C})$, i.e. X commutes with the entries of $[L]_{11}$, $[L]_{22}$, $[L]_{33}$. We will show that X is sufficient to break the multiplicity of $V^{(3,2,1,1,0)}$ in $W^{(2,1,2,1,1)}$. In order to diagonalize X on $W^{(2,1,2,1,1)}$ we diagonalize it first on $P^{(2,1,2,1,1)}$ and then project into $W^{(2,1,2,1,1)}$ using the operators L_{12} and L_{34}. This is valid since $[X, L_{12}] = [X, L_{34}] = 0$. So we calculate

$[X, \Lambda_i]f$, $i = 1, \ldots, 6$, using the procedure outlined in the proof of theorem 3.3. These calculations yield

$$
\begin{aligned}
[X, \Lambda_1]f &= (-2\Lambda_1 + 8\Lambda_7)f \\
[X, \Lambda_2]f &= (-2\Lambda_2 - 4\Lambda_3 - 12\Lambda_6 - 8\Lambda_7)f \\
[X, \Lambda_3]f &= (-2\Lambda_1 - 2\Lambda_2 - 4\Lambda_3 - 12\Lambda_6 - 6\Lambda_7)f \\
[X, \Lambda_4]f &= (2\Lambda_2 - 2\Lambda_3 - 6\Lambda_4 - 2\Lambda_7 - 2\Lambda_8 - 4\Lambda_{10})f \\
[X, \Lambda_5]f &= (2\Lambda_2 - 8\Lambda_4 - 22\Lambda_5 - 8\Lambda_6 - 4\Lambda_7 - 4\Lambda_8 - 6\Lambda_9 - 12\Lambda_{10})f \\
[X, \Lambda_6]f &= (-6\Lambda_1 - 14\Lambda_2 - 10\Lambda_3 + 6\Lambda_6 - 2\Lambda_7)f
\end{aligned}
$$

where $\Lambda_7 = L_{21}L_{32}L_{42}L_{54}$, $\Lambda_8 = L_{21}L_{32}L_{32}L_{53}$, $\Lambda_9 = L_{31}L_{32}L_{53}$, $\Lambda_{10} = L_{21}L_{32}L_{52}$. Next we calculate $[X, \Lambda_i]f$, $i = 7, \ldots, 10$, to see if the procedure closes on itself. These calculations yield

$$
\begin{aligned}
[X, \Lambda_7]f &= (4\Lambda_1 + 2\Lambda_7)f \\
[X, \Lambda_8]f &= (-4\Lambda_1 - 8\Lambda_7 - 6\Lambda_8)f \\
[X, \Lambda_9]f &= (-4\Lambda_2 + 4\Lambda_3 + 12\Lambda_6 + 8\Lambda_7 - 6\Lambda_9)f \\
[X, \Lambda_{10}]f &= (2\Lambda_1 + 4\Lambda_7 + 6\Lambda_8 + 6\Lambda_{10})f.
\end{aligned}
$$

Since this calculation does not yield any new operators, our process has closed upon itself. Eventually, the procedure will always close upon itself since there are only a finite collection of maps form $V^{(m)}$ into $P^{(M)}$ which consist solely of lowering operators.

Thus, we must diagonalize the matrix:

$$
C = \begin{pmatrix}
-2 & 0 & -2 & 0 & 0 & -6 & 4 & -4 & 0 & 2 \\
0 & -2 & -2 & 2 & 2 & -14 & 0 & 0 & -4 & 0 \\
0 & -4 & -4 & -2 & 0 & -10 & 0 & 0 & 4 & 0 \\
0 & 0 & 0 & -6 & -8 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -22 & 0 & 0 & 0 & 0 & 0 \\
0 & -12 & -12 & 0 & -8 & 6 & 0 & 0 & 12 & 0 \\
8 & -8 & -6 & -2 & -4 & -2 & 2 & -8 & 8 & 4 \\
0 & 0 & 0 & -2 & -4 & 0 & 0 & -6 & 0 & 6 \\
0 & 0 & 0 & 0 & -6 & 0 & 0 & 0 & -6 & 0 \\
0 & 0 & 0 & -4 & -12 & 0 & 0 & 0 & 0 & 6
\end{pmatrix}.
$$

The eigenvalues of the matrix C are: $\lambda_1 = -22$, $\lambda_2 = 0$, $\lambda_3 = 18$, $\lambda_4 = -18$, $\lambda_5, \lambda_6 = 6$, $\lambda_7, \lambda_8, \lambda_9, \lambda_{10} = -6$. The eigenvectors of C which give our shift operators are:

$\lambda = -22$ $\quad \tilde{\Lambda}_1 = \Lambda_1 + 3\Lambda_2 + 4\Lambda_3 + 20\Lambda_4 + 40\Lambda_5 + 8\Lambda_6 + 4\Lambda_7 + 5\Lambda_8 + 15\Lambda_9 + 20\Lambda_{10}$

$\lambda = 0$ $\quad \tilde{\Lambda}_2 = \Lambda_1 + 3\Lambda_2 - 3\Lambda_3 - \Lambda_7$

$\lambda = 18$ $\quad \tilde{\Lambda}_3 = 2\Lambda_1 + 6\Lambda_2 + 3\Lambda_3 - 9\Lambda_6 - 2\Lambda_7$

$\lambda = -18$ $\quad \tilde{\Lambda}_4 = \Lambda_1 + 3\Lambda_2 + 3\Lambda_3 + 3\Lambda_6 + 2\Lambda_7$

$$\lambda = 6 \qquad \tilde{\Lambda}_5 = \Lambda_1 + 2\Lambda_7$$

$$\tilde{\Lambda}_6 = \Lambda_8 + 2\Lambda_{10}$$

$$\lambda = -6 \qquad \tilde{\Lambda}_7 = -\Lambda_1 - 3\Lambda_2 + 3\Lambda_3 + 9\Lambda_4 + \Lambda_7 + 3\Lambda_{10}$$

$$\tilde{\Lambda}_8 = \Lambda_1 - \Lambda_7$$

$$\tilde{\Lambda}_9 = \Lambda_1 + \Lambda_8$$

$$\tilde{\Lambda}_{10} = \Lambda_2 + \Lambda_9.$$

To find how $\Lambda_7 f, \ldots, \Lambda_{10} f$ depend on $\Lambda_1 f, \ldots, \Lambda_6 f$, we must choose $f \in V^{(3,2,1,1)}$ and form $\tilde{\Lambda}_1 f, \ldots, \tilde{\Lambda}_{10} f$ explicitly. Let $f = \Delta_1^1 \Delta_{12}^{12} \Delta_{123}^{123} \Delta_{1234}^{1234}$, then f is the highest weight vector of $V^{(3,2,1,1)}$ (see [7]). Generating the polynomials $\tilde{\Lambda}_1 f, \ldots, \tilde{\Lambda}_{10} f$, we find that $\tilde{\Lambda}_5 f = \tilde{\Lambda}_6 f = \tilde{\Lambda}_9 f = \tilde{\Lambda}_{10} f = 0$ which tells us:

$$\Lambda_7 f = -\tfrac{1}{2}\Lambda_1 f$$

$$\Lambda_8 f = -\Lambda_1 f$$

$$\Lambda_9 f = -\Lambda_2 f$$

$$\Lambda_{10} f = \tfrac{1}{2}\Lambda_1 f.$$

Thus $\{\Lambda_1, \ldots, \Lambda_6\}$ is a basis for $\mathrm{Hom}_{\mathrm{GL}(4,\mathbb{C})}(V^{(3,2,1,1,0)}, P^{(2,1,2,1,1)})$. Therefore, the six shift operators which send $V^{(3,2,1,1)}$ into $P^{(2,1,2,1,1)}$ are

$$\lambda = -22 \qquad \tilde{\Lambda}_1 = \Lambda_1 - 3\Lambda_2 + \Lambda_3 + 5\Lambda_4 + 10\Lambda_5 + 2\Lambda_6$$

$$\lambda = 0 \qquad \tilde{\Lambda}_2 = \Lambda_1 + 2\Lambda_2 - 2\Lambda_3$$

$$\lambda = 18 \qquad \tilde{\Lambda}_3 = \Lambda_1 + 2\Lambda_2 + \Lambda_3 - 3\Lambda_6$$

$$\lambda = -18 \qquad \tilde{\Lambda}_4 = \Lambda_2 + \Lambda_3 + \Lambda_6$$

$$\lambda = -6 \qquad \tilde{\Lambda}_5 = \Lambda_2 - \Lambda_3 - 3\Lambda_4$$

$$\lambda = -6 \qquad \tilde{\Lambda}_6 = \Lambda_1.$$

It remains to find which of the eigenvectors of $X, \tilde{\Lambda}_1 f, \ldots, \tilde{\Lambda}_6 f$ are simultaneously annihilated by L_{12} and L_{34}. Using the polynomials found earlier, we operate L_{12} and L_{34} on each other and find that $\tilde{\Lambda}_1 f, \tilde{\Lambda}_2 f, \tilde{\Lambda}_3 f$, and $\tilde{\Lambda}_5 f$ are simultaneously annihilated. Therefore, the multiplicity of $V^{(3,2,1,1)}$ in $W(2,1,2,1,1)$ is four. Since the eigenvalues corresponding to the polynomials that were simultaneously annihilated by X are distinct, and X is Hermitian, the eigenvectors are obviously orthogonal. In conclusion, the four intertwining operators that send $V^{(3,2,1,1)}$ into four orthogonal (equivalent) submodules of $W^{(2,1,2,1,1)}$ are:

$$\tilde{\Lambda}_1 = L_{21}L_{32}^2L_{43}L_{54} - 3L_{31}L_{32}L_{43}L_{54} + L_{31}L_{42}L_{54} + 5L_{31}L_{52} + 10L_{32}L_{51} + 2L_{32}L_{41}L_{54}$$

$$\tilde{\Lambda}_2 = L_{21}L_{32}^2L_{43}L_{54} + 2L_{31}L_{32}L_{43}L_{54} - 2L_{31}L_{42}L_{54}$$

$$\tilde{\Lambda}_3 = L_{21}L_{32}^2L_{43}L_{54} + 2L_{31}L_{32}L_{43}L_{54} + L_{31}L_{42}L_{54} - 3L_{32}L_{41}L_{54}$$

$$\tilde{\Lambda}_4 = L_{31}L_{32}L_{43}L_{54} - L_{31}L_{42}L_{54} - 3L_{31}L_{52}$$

and any element $f \in V^{(3,2,1,1)}$ can be mapped with these operators into $W^{(2,1,2,1,1)}$. In particular, if basis elements in $V^{(3,2,1,1)}$ and $W^{(2,1,2,1,1)}$ are chosen, it is straightforward to compute the Clebsch–Gordan coefficients relative to this basis choice. Similarly Racah coefficients (which are already basis independent) can easily be computed; however, in this case the invariant operators are determined by the coupling scheme that is chosen.

5. Conclusion

We have shown how to decompose an r-fold tensor product of arbitrary irreducible representations using shift operators. Underlying our procedure is the use of polynomial realizations of all the reducible representations of the unitary groups. These polynomial realizations have the advantage of being basis independent; different bases, dictated by physical considerations, result in different sets of polynomials, and the transformation coefficients between the basis sets are easily calculated using the differentiation inner product (2.2) defined in section 2. We are not interested in finding closed-form expressions for Clebsch–Gordan or Racah coefficients but instead have given well-defined procedures that can be adapted for the computer. We shall now briefly describe our procedures for generating the shift operators for a given tensor product of arbitrary irreducible representations of U(N).

We assume that an r-fold tensor product of irreducible representations with signatures $(M_{(1)}), \ldots, (M_{(r)})$ is given; the goal is to give an orthogonal direct sum decomposition of the r-fold tensor product into irreducible representations of U(N). This is equivalent to finding *shift operators which map the irreducible representation space $V^{(m)}$ into an orthogonal direct* sum of $\mathcal{I}(V^{(m)}) \cap H^{(M)}$. We begin by forming an n-tuple of integers from the entries of $(M_{(1)}), \ldots, (M_{(r)})$ by deleting the non-zero entries from each $(M_{(i)})$. Next we introduce the Fock space $\mathcal{F}(\mathbb{C}^{n\times N})$ and define an action R of U(N) on $\mathcal{F}$ by right translation. In fact, we only need to consider a finite-dimensional subspace of $\mathcal{F}(\mathbb{C}^{n\times N})$, namely $P^{(M)}$, which consists of polynomials which transform covariantly with respect to the diagonal subgroup $D_n \subset \mathrm{GL}(n, \mathbb{C})$. By theorem 2.1 the number of times that $V^{(m)}$ occurs in $P^{(M)}$ is equal to the dimension of the weight space (M) in $V^{(m)}$. This multiplicity is calculated by generating the set of all Gelfand tableaux $\binom{(m)}{[t]}$ with weight (M). We then use these tableaux to construct a basis for $\mathrm{Hom}_H(V^{(m)}, P^{(M)})$. The space $W^{(M)}$ which is isomorphic to the r-fold tensor product $V^{(M_{(1)})} \otimes \cdots \otimes (M_{(r)})$ is defined to be the set of all polynomials in $P^{(M)}$ which are simultaneously annihilated by the infinitesimal raising operators of G', $L_{\alpha_{p_i}\beta_{p_i}}$. We choose a commuting family of invariant operators $X_1, \ldots, X_q$ in $C_{U(H')}(G')$ whose eigenvalues can be used to break the multiplicity of $V^{(m)}$ in $W^{(M)}$. To break this multiplicity we construct shift operators $\tilde{\Lambda}_1, \ldots, \tilde{\Lambda}_\mu$ which satisfy $[X_i \tilde{\Lambda}_j]f = \lambda_{ij}(m)\tilde{\Lambda}_j f$, $i = 1, \ldots, q$, $j = 1, \ldots, \mu$. To construct these shift operators we use the algorithm described in the proof of theorem 3.1 to generate q-simultaneously diagonalizable matrices whose eigenvalues are used to break the multiplicity and whose eigenvectors generate the shift operators. It should be noted that our method of diagonalizing these invariant operators depends only on the commutation relations of the Lie algebra generated by the infinitesimal operators of H' and the action of these operators on the given irreducible representation space $V^{(m)}$. Finally we form the polynomials $\tilde{\Lambda}_1 f. \ldots, \tilde{\Lambda}_\mu F$, where $f \in V^{(m)}$, and explicitly calculate $L_{\alpha_{p_i}\beta_{p_i}}\tilde{\Lambda}_k f$ $i = 1, \ldots, r$, $k = 1, \ldots, \mu$.

Those polynomials which are simultaneously diagonalized by all the $L_{\alpha_{p_i}\beta_{p_i}}$ then give us an orthogonal direct sum decomposition of $\mathcal{I}(V^{(m)}) \cap W^{(M)}$.

There are still a number of problems associated with the tensor product decomposition of arbitrary irreducible representations of U(N). In this paper we have shown how the multiplicity problem may be stated for arbitrary group–subgroup pairs and then applied this formalism to the decomposition of tensor products. In particular, we have shown that shift operators always exist for this problem and given an outline of how these shift operators can be generated. In a forthcoming paper we will discuss the computational aspects of this problem as well as show how to deal with irrational eigenvalues of our invariant operators [8]. We also plan to generalize Hughes's problem, that is to investigate the multiplicity problem for U(N) restricted to SO(N) and to Sp(N).

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